Lösungen A7-A13

A7

a) $\int_0^{\pi/2} 3\sin(x)\cos(x)\,dx \qquad u = \sin(x),\ \frac{du}{dx} = \cos(x),\ dx = \frac{du}{\cos(x)}$

$$\| $$

$$\int_{u(0)}^{u(\pi/2)} 3u\,du = \left[\frac{3}{2}u^2\right]_{\sin(0)}^{\sin(\frac{\pi}{2})} = \left[\frac{3}{2}u^2\right]_0^1 = \underline{\frac{3}{2}}$$

b) $\int_0^{14} \frac{1}{4+x^2}\cdot 2x\,dx \qquad u = 4+x^2,\ \frac{du}{dx} = 2x,\ dx = \frac{du}{2x}$

$$\| $$

$$\int_{u(0)}^{u(14)} \frac{1}{u}\,du = \left[\ln(u)\right]_4^{200} = \ln(u) - \ln(4) = \ln\left(\frac{200}{4}\right) = \underline{\ln(50)}$$

c) $\int_{-1}^{2} x(1+x^2)^3\,dx \qquad u = 1+x^2,\ \frac{du}{dx} = 2x,\ dx = \frac{du}{2x}$

$$\| $$

$$\int_{u(-1)}^{u(2)} \frac{1}{2}u^3\,du = \left[\frac{1}{8}u^4\right]_2^5 = \frac{1}{8}\left(5^4 - 2^4\right) = \frac{1}{8}(625-16) = \underline{\frac{609}{8}}$$

d) $\int_1^e x^3\ln(x^4) \qquad u = x^4,\ \frac{du}{dx} = 4x^3,\ dx = \frac{du}{4x^3}$

$$\| $$

$$\int_{u(1)}^{u(e)} \frac{1}{4}\ln(u)\,du = \left[\frac{1}{4}\left(u\cdot\ln(u) - u\right)\right]_1^{e^4} = \frac{1}{4}\left(e^4\cdot 4 - e^4 - (0-1)\right) = \frac{1}{4}\left(3\cdot e^4 + 1\right)$$

$$= \underline{\frac{3}{4}e^4 + \frac{1}{4}}$$

e) $\int_0^4 \frac{x}{\sqrt{9+x^2}}\,dx \qquad u = 9+x^2,\ \frac{du}{dx} = 2x,\ dx = \frac{du}{2x}$

$$\| $$

$$\int_{u(0)}^{u(4)} \frac{1}{2\sqrt{u}}\,du = \left[\sqrt{u}\right]_9^{25} = 5 - 3 = \underline{2}$$

f) $\int_2^4 \sqrt{x^2(20-x^2)}\,dx \qquad u = 20-x^2,\ \frac{du}{dx} = -2x,\ dx = \frac{du}{-2x}$

$$\| $$

$$\int_{u(2)}^{u(4)} \frac{1}{-2}\sqrt{u}\,du = \left[-\frac{1}{2}\,\frac{2}{3}\,u^{\frac{3}{2}}\right]_{16}^4 = -\frac{1}{3}4^{\frac{3}{2}} + \frac{1}{3}16^{\frac{3}{2}} = -\frac{8}{3} + \frac{64}{3} = \underline{\frac{56}{3}}$$

A8

a) $\int_e^{e^2} \frac{1}{x}\ln(x)\,dx \qquad$ Substitution: $u = \ln(x),\ \frac{du}{dx} = \frac{1}{x},\ dx = x\cdot du$

$$\| $$

$$\int_{u(e)}^{u(e^2)} u\,du = \left[\frac{1}{2}u^2\right]_1^2 = \frac{1}{2}4 - \frac{1}{2} = \frac{3}{2}$$

Partielle Integration

Partielle Integration

$$\underbrace{\int_e^{e^2} \underset{v'}{\frac{1}{x}} \underset{u}{\ln(x)}\, dx}_{A} = \Big[\underset{v}{\ln(x)} \cdot \underset{u}{\ln(x)}\Big]_e^{e^2} - \underbrace{\int_e^{e^2} \underset{v}{\ln(x)} \cdot \underset{u'}{\frac{1}{x}}\, dx}_{A}$$

$$= \frac{1}{2}\left(\ln(e^2)^2 - \ln(e)^2\right) = \frac{1}{2}(4-1) = \underline{\frac{3}{2}}$$

b) $$\underbrace{\int_{-\pi/2}^{\pi/2} \underbrace{\tfrac{1}{2} \sin^2 x}_{u} \cdot \underbrace{\cos x}_{v'}\, dx}_{A} = \Big[\underset{u}{\tfrac{1}{2} \sin^2 x} \cdot \underset{v}{\sin x}\Big]_{-\pi/2}^{\pi/2} - \underbrace{\int_{-\pi/2}^{\pi/2} \underbrace{\sin x \cdot \cos x}_{u'} \cdot \underset{v}{\sin x}\, dx}_{2A}$$

$$= \frac{1}{3} \cdot \frac{1}{2}\Big[\sin^3 x\Big]_{-\pi/2}^{\pi/2} = \frac{1}{6}\left(1^3 - (-1)^3\right) = \underline{\frac{1}{3}}$$

Substitution: $u = \sin x, \quad \frac{du}{dx} = \cos x, \quad dx = \frac{du}{\cos x}$

$$\int_{-\pi/2}^{\pi/2} \frac{1}{2}\sin^2 x \cdot \cos x\, dx = \int_{u(-\pi/2)}^{u(\pi/2)} \frac{1}{2} u^2\, du = \Big[\frac{1}{6}u^3\Big]_{-1}^{1} = \frac{1}{6} + \frac{1}{6} = \underline{\frac{1}{3}}$$

A9

a) $$\int_0^1 \frac{1}{1+\sqrt{x}}\, dx \qquad u = 1+\sqrt{x}, \quad \frac{du}{dx} = \frac{1}{2\sqrt{x}}, \quad dx = 2\sqrt{x}\, du$$

$$\overset{\shortparallel}{=} \int_{u(0)}^{u(1)} \frac{2\sqrt{x}}{u}\, du = \int_{u(0)}^{u(1)} \frac{2(u-1)}{u}\, du = \int_1^2 2 - \frac{2}{u}\, du = \Big[2u - 2\cdot\ln|u|\Big]_1^2$$

$$= 4 - 2\cdot\ln 2 - (2-0) = \underline{2 - 2\cdot\ln 2}$$

b) $$\int_1^2 \frac{x+1}{x^2+4x+4}\, dx = \int_1^2 \frac{x+1}{(x+2)^2}\, dx \qquad u = x+2, \quad \frac{du}{dx} = 1, \quad dx = du$$

$$= \int_{u(1)}^{u(2)} \frac{u-1}{u^2}\, dx = \int_3^4 \frac{1}{u} - \frac{1}{u^2}\, du = \Big[\ln|u| + \frac{1}{u}\Big]_3^4$$

$$= \ln 4 + \frac{1}{4} - \left(\ln 3 + \frac{1}{3}\right) = \underline{\ln\frac{4}{3} - \frac{1}{12}}$$

c) $$\int_1^{16} \frac{6}{2+\sqrt{x}}\, dx \qquad u = 2+\sqrt{x}, \quad \frac{du}{dx} = \frac{1}{2\sqrt{x}}, \quad dx = 2\sqrt{x}\, du$$

$$\overset{\shortparallel}{=} \int_{u(1)}^{u(16)} \frac{6\cdot 2\sqrt{x}}{u}\, du = \int_3^6 \frac{6\cdot 2(u-2)}{u}\, du = \int_3^6 12 - \frac{24}{u}\, du = \Big[12u - 24\cdot\ln|u|\Big]_3^6$$

$$= 72 - 24\ln 6 - 36 + 24\ln 3 = \underline{36 - 24\ln 2}$$

d) $$\int_0^{\sqrt{5}} \frac{x^3}{\sqrt{9-x^2}}\, dx \qquad u = 9-x^2, \quad \frac{du}{dx} = -2x, \quad dx = -\frac{du}{2x}$$

$$\overset{\shortparallel}{=} \int_{u(0)}^{u(\sqrt{5})} -\frac{x^3}{\sqrt{u}\, 2x}\, du = \int_9^4 -\frac{x^2}{2\sqrt{u}}\, du = \int_9^4 \frac{u-9}{2\sqrt{u}}\, du = \int_9^4 \frac{\sqrt{u}}{2} - \frac{9}{2\sqrt{u}}\, du$$

$$= \Big[\frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}} - 2\cdot\frac{9}{2}\sqrt{u}\Big]_9^4 = \left(\frac{1}{3}\cdot 8 - 18 - (9-27)\right) = \underline{\frac{8}{3}}$$

A10

$\int x \ldots \quad u = 4-x^2 \quad \ldots \quad dx = \frac{du}{\ldots}$

A10

a) $\int \frac{x}{\sqrt{4-x^2}}\,dx \qquad u = 4-x^2,\ \frac{du}{dx} = -2x,\ dx = \frac{du}{-2x}$

$$\int \frac{x}{\sqrt{u}} \cdot \frac{du}{-2x} = \int -\frac{1}{2\sqrt{u}}\,du = -\sqrt{u} = -\sqrt{4-x^2}$$

$$\underline{F(x) = -\sqrt{4-x^2}}$$

b) $\int \frac{e^{2x}}{(e^x-2)^3}\,dx \qquad u = e^x - 2,\ \frac{du}{dx} = e^x,\ dx = \frac{du}{e^x}$

$$\int \frac{e^{2x}}{u^3}\,\frac{du}{e^x} = \int \frac{e^x}{u^3}\,du = \int \frac{u+2}{u^3}\,du = \int \frac{1}{u^2} + \frac{2}{u^3}\,du$$

$$= -\frac{1}{u} - \frac{2}{2u^2} = -\frac{1}{e^x-2} - \frac{1}{(e^x-2)^2} = \frac{-e^x+1}{(e^x-2)^2}$$

$$\underline{F(x) = \frac{-e^x+1}{(e^x-2)^2}}$$

A11

a) $\frac{3x+3}{(x-2)(x+7)} = \frac{A}{(x-2)} + \frac{B}{(x+7)} \qquad A = \frac{3\cdot 2+3}{2+7} = 1,\ B = \frac{3(-7)+3}{-9} = 2$

$$\int_{-1}^{1} \frac{3x+3}{(x-2)(x+7)}\,dx = \int_{-1}^{1} \frac{1}{(x-2)} + \frac{2}{(x+7)}\,dx = \Big[\ln|x-2| + 2\ln|x+7|\Big]_{-1}^{1}$$

$$= \ln 1 + 2\ln 8 - (\ln 3 + 2\ln 6) = 0 + 2\cdot\ln 2^3 - \ln 3 - 2\ln(2\cdot 3)$$

$$= 6\ln 2 - \ln 3 - 2(\ln 2 + \ln 3) = \underline{4\ln 2 - 3\ln 3}$$

b) $\frac{2}{(x-4)(x+1)} = \frac{A}{(x-4)} + \frac{B}{(x+1)} \qquad A = \frac{2}{4+1} = \frac{2}{5},\ B = \frac{2}{-1-4} = -\frac{2}{5}$

$$\int_{5}^{6} \frac{2}{(x-4)(x+1)}\,dx = \int_{5}^{6} \frac{2}{5(x-4)} - \frac{2}{5(x+1)}\,dx = \Big[\tfrac{2}{5}\ln|x-4| - \tfrac{2}{5}\ln|x+1|\Big]_{5}^{6} = \frac{2}{5}\big(\ln 2 - \ln 7 - (\ln 1 - \ln 6)\big)$$

$$= \underline{\frac{2}{5}\left(2\ln 2 + \ln 3 - \ln 7\right)}$$

c) $\frac{2x+2}{x(x-1)(x-2)} = \frac{A}{x} + \frac{B}{(x-1)} + \frac{C}{(x-2)} \qquad A = \frac{0+2}{-1\cdot(-2)} = 1$

$$B = \frac{2\cdot 1+2}{1(1-2)} = \frac{4}{-1} = -4,\quad C = \frac{2\cdot 2+2}{2\cdot(2-1)} = \frac{6}{2} = 3$$

$$\int_{3}^{4} \frac{2x+2}{x(x-1)(x-2)}\,dx = \int_{3}^{4} \frac{1}{x} - \frac{4}{x-1} + \frac{3}{x-2}\,dx = \Big[\ln|x| - 4\ln|x-1| + 3\ln|x-2|\Big]_{3}^{4}$$

$$= \ln 4 - 4\ln 3 + 3\ln 2 - (\ln 3 - 4\ln 2 + 3\ln 1)$$

$$= 2\ln 2 - 4\ln 3 + 3\ln 2 - \ln 3 + 4\ln 2 = \underline{9\ln 2 - 5\ln 3}$$

(A12)

a) $\frac{2x-4}{(x-3)^2} = \frac{A}{(x-3)} + \frac{B}{(x-3)^2}$, $B = 2\cdot 3 - 4 = 2$

$A(x-3) + B = 2x \Rightarrow A = 2$

$$\int_0^1 \frac{2x-4}{(x-3)^2}\,dx = \int_0^1 \frac{2}{(x-3)} + \frac{2}{(x-3)^2}\,dx = \left[2\ln|x-3| - \frac{2}{(x-3)}\right]_0^1 = 2\ln 2 - \frac{2}{-2} - \left(2\ln 3 + \frac{2}{3}\right)$$

$$= 2\ln 2 + 1 - 2\ln 3 - \frac{2}{3} = \underline{2\ln 2 - 2\ln 3 + \frac{1}{3}}$$

b) $x^2 - 3x - 10 = 0$

$x_{1/2} = \frac{3 \pm \sqrt{9+40}}{2} = \frac{3 \pm 7}{2}$ $\quad x_1 = 5,\ x_2 = -2$

$\frac{7x+7}{(x-5)(x+2)} = \frac{A}{(x-5)} + \frac{B}{(x+2)}$ $\quad A = \frac{7\cdot 5 + 7}{7} = 6,\ B = \frac{7\cdot(-2)+7}{-7} = 1$

$$\int_1^2 \frac{7x+7}{x^2-3x-10}\,dx = \int_1^2 \frac{6}{x-5} + \frac{1}{x+2}\,dx = \left[6\ln|x-5| + \ln|x+2|\right]_1^2 = 6\ln 3 + \ln 4 - (6\ln 4 + \ln 3)$$

$$= \underline{5\ln 3 - 5\ln 4}$$

c) $x^3 - 3x - 2 \quad x_1 = -1$

$(x^3 - 3x - 2) : (x+1) = x^2 - x - 2$

$\underline{x^3 + x^2}$

$-x^2 - 3x - 2$

$\underline{-x^2 - x}$

$-2x - 2$

$x^2 - x - 2 = 0$

$x_{2/3} = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2}$

$x_2 = 2,\ x_3 = -1$

Also: $(x^3 - 3x - 2) = (x+1)^2(x-2)$

$\frac{9x^2 + 9x + 9}{(x+1)^2(x-2)} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x-2)}$

$B = \frac{9-9+9}{-3} = -3$, $C = \frac{9\cdot 4 + 9\cdot 2 + 9}{(2+1)^2} = 7$

A (Betrachte Koeffizient von x^2): $9 = A + C \Rightarrow A = 2$

$$\int_0^1 \frac{9x^2+9x+9}{x^3-3x-2}\,dx = \int_0^1 \frac{2}{x+1} - \frac{3}{(x+1)^2} + \frac{7}{(x-2)}\,dx$$

$$= \left[2\ln|x+1| + \frac{3}{x+1} + 7\ln|x-2|\right]_0^1 = 2\ln 2 + \frac{3}{2} + 7\ln 1 - (2\ln 1 + 3 + 7\ln 2)$$

$$= 2\ln 2 + \frac{3}{2} - 3 - 7\ln 2 = \underline{-5\ln 2 - \frac{3}{2}}$$

d) $(x^2+4) : (x^2-4) = 1 + \frac{8}{x^2-4}$

$\underline{x^2 - 4}$

8

$\frac{8}{(x+2)(x-2)} = \frac{A}{(x-2)} + \frac{B}{(x+2)}$ $\quad A = \frac{8}{4} = 2,\ B = \frac{8}{-4} = -2$

$$\int_3^5 \frac{x^2+4}{x^2-4}\,dx = \int_3^5 1 + \frac{2}{x-2} - \frac{2}{x+2}\,dx = \left[x + 2\ln|x-2| - 2\ln|x+2|\right]_3^5$$

$$\int_3 \frac{x^2+4}{x^2-4}\,dx = \int_3 1+\frac{2}{x-2}-\frac{2}{x+2}\,dx = \Big[x+2\ln|x-2|-2\ln|x+2|\Big]_3$$

$$= 5+2\ln 3-2\ln 7-\big(3+2\ln 1-2\ln 5\big) = 2+2\ln 3-2\ln 7+2\ln 5$$

$$= 2+2\ln\left(\tfrac{3\cdot 5}{7}\right) = \underline{2+2\ln\tfrac{15}{7}}$$

e) $(4x^2-2x+2):(x^2-4x+3) = 4+\dfrac{14x-10}{x^2-4x+3}$

$\quad \underline{4x^2-16x+12}$

$\quad 14x-10$

$$x^2-4x+3=0$$
$$x_{1/2} = \frac{4\pm\sqrt{16-12}}{2} = \frac{4\pm 2}{2}$$
$$x_1 = 3,\ x_2 = 1$$

$$\frac{14x-10}{(x-3)(x-1)} = \frac{A}{(x-3)}+\frac{B}{(x-1)} \qquad A=\frac{14\cdot 3-10}{2}=16,\quad B=\frac{14\cdot 1-10}{-2}=-2$$

$$\int_{-1}^{0}\frac{4x^2-2x+2}{x^2-4x+3}\,dx = \int_{-1}^{0} 4+\frac{16}{(x-3)}-\frac{2}{(x-1)}\,dx = \Big[4x+16\ln|x-3|-2\ln|x-1|\Big]_{-1}^{0}$$

$$= 16\ln 3-2\ln 1-\big(-4+16\ln 4-2\ln 2\big) = \underline{4-30\ln 2+16\ln 3}$$

A13

a) $\displaystyle\int_0^{1/2}\frac{9x+3}{x^2-1}\,dx$ Partialbruchzerlegung

$$\frac{9x+3}{(x+1)(x-1)} = \frac{A}{(x+1)}+\frac{B}{(x-1)} \qquad A=\frac{9(-1)+3}{-2}=3,\quad B=\frac{9\cdot 1+3}{2}=6$$

$$\int_0^{1/2}\frac{9x+3}{x^2+1}\,dx = \int_0^{1/2}\frac{3}{x+1}+\frac{6}{x-1}\,dx = \Big[3\ln|x+1|+6\ln|x-1|\Big]_0^{1/2} = 3\ln\tfrac{3}{2}+6\ln\tfrac{1}{2}-\big(3\ln 1+6\ln 1\big)$$

$$= 3(\ln 3-\ln 2)+6(\ln 1-\ln 2) = \underline{3\ln 3-9\ln 2}$$

b) $\displaystyle\int_0^9\frac{\sqrt{x}}{4+\sqrt{x}}\,dx \qquad u=4+\sqrt{x},\quad \frac{du}{dx}=\frac{1}{2\sqrt{x}},\quad dx=2\sqrt{x}\,du$

$$\int_{u(0)}^{u(9)}\frac{\sqrt{x}}{u}\,2\sqrt{x}\,du = \int_4^7\frac{2x}{u}\,du = \int_4^7\frac{2(u-4)^2}{u}\,du = \int_4^7\frac{2u^2-16u+32}{u}\,du = \int_4^7 2u-16+\frac{32}{u}\,du$$

$$= \Big[u^2-16u+32\ln|u|\Big]_4^7 = 49-112+32\ln 7-16+64-32\ln 4 = \underline{-15+32\ln 7-64\ln 2}$$

c) $$\int_{-\pi}^{\pi}\underbrace{(x+3)}_{u}\,\underbrace{\sin x}_{v'}\,dx = \Big[\underset{u}{(x+3)}\,\underset{v}{(-\cos x)}\Big]_{-\pi}^{\pi}-\int_{-\pi}^{\pi}\underset{u'}{1}\cdot\underset{v}{(-\cos x)}\,dx$$

$$= (\pi+3)\cdot 1-\big((-\pi+3)\cdot 1\big)+\Big[\sin x\Big]_{-\pi}^{\pi} = \pi+3+\pi-3 = \underline{2\pi}$$

d) $\displaystyle\int^{\ln 5}\frac{e^x-1}{\dots}\,dx \qquad u=e^x+1 \qquad \frac{du}{dx}=e^x \qquad dx=\frac{du}{e^x}$

$$d)\quad \int_0^{\ln 5} \frac{e^x - 1}{e^x + 1}\,dx \qquad u = e^x + 1 \qquad \frac{du}{dx} = e^x \qquad dx = \frac{du}{e^x}$$

$$\shortparallel$$

$$\int_{u(0)}^{u(\ln(5))} \frac{e^x - 1}{u} \cdot \frac{du}{e^x} = \int_2^6 \frac{u-2}{u\cdot(u-1)}\,du \quad \Big\} \text{ Partialbruchzerlegung}$$

$$\frac{u-2}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1}\ , \quad A = \frac{0-2}{-1} = 2\ , \quad B = \frac{1-2}{1} = -1$$

$$\int_2^6 \frac{u-2}{u\cdot(u-1)}\,du = \int_2^6 \frac{2}{u} - \frac{1}{u-1}\,du = \Big[2\ln|u| - \ln|u-1|\Big]_2^6 = 2\ln 6 - \ln 5 - (2\ln 2 - \ln 1)$$

$$= 2(\ln 2 + \ln 3) - \ln 5 - 2\ln 2 = \underline{2\ln 3 - \ln 5}$$